Modal Propositional Logic

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Modal propositional logic: Syntax

- Syntax of classical propositional logic
- Countable set of propositional atoms: $p_0, p_1 \ldots, p_n, \ldots$
- Logical connectives: $\wedge, \vee, \rightarrow, \neg$
- Modal operators: $\Box, \Diamond$ (necessity, possibility)
- If α is a wff, so is $\Box\alpha$. $\Diamond\alpha =_{df} \neg\Box\neg\alpha$

Normal Modal Systems

- A *system of modal logic* is a certain class S of formulas whose elements are *theorems*
- $\vdash_S \alpha$ denotes that α is a theorem of S.
- A modal system is *normal* if it contains:
- all theorems of propositional logic
- the axiom $\mathbf{K}$: $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$
- the rules US (uniform substitution): if α is a theorem so is any substitution instance
- MP (modus ponens)
- N (necessitation): $\vdash \alpha \Rightarrow \vdash \Box\alpha$

Models of normal systems

A model is a triple $\langle W, R, V \rangle$, where

- W is a non-empty set (of possible worlds)
- R is a binary relation over W, ie $R \subseteq W \times W$
- V is a valuation assigning a truth-value 1 or 0 to each atomic proposition p at each world $w \in W$

Models of normal systems, contd

Valuations V are extended to all formulas via the following rules:

(V $\wedge$) $V(\alpha \wedge \beta, w) = 1$ iff $V(\alpha, w) = 1$ & $V(\beta, w) = 1$

(V $\vee$) $V(\alpha \vee \beta, w) = 1$ iff $V(\alpha, w) = 1$ or $V(\beta, w) = 1$

(V $\rightarrow$) $V(\alpha \rightarrow \beta, w) = 1$ iff $V(\alpha, w) = 1$ implies $V(\beta, w) = 1$

(V $\neg$) $V(\neg\alpha, w) = 1$ iff $V(\alpha, w) = 0$

(V $\Box$) $V(\Box\alpha, w) = 1$ iff $V(\alpha, w') = 1$, for all w' such that wRw'

Truth and Validity

- A formula α is true at world w in a model $\mathcal{M} = \langle W, R, V \rangle$ if $V(\alpha, w) = 1$. In this case we sometimes write $\mathcal{M}, w \models \alpha$.

- A formula α is true in a model $\langle W, R, V \rangle$ if $V(\alpha, w) = 1$, for all $w \in W$. We also write $\mathcal{M} \models \alpha$.

- A formula is valid (in a class of models) if it is true in every model in that class

- The axiom $\mathbf{K}$ is valid in the class of all models

- The weakest normal system axiomatised by $\mathbf{K}$ and propositional logic is called K. Its theorems are true in all models

some other axioms of normal systems

Stronger normal systems can be obtained by adding further axioms

$\mathbf{T}: \Box p \rightarrow p$

$\mathbf{D}: \Box p \rightarrow \Diamond p$

$\mathbf{4}: \Box p \rightarrow \Box\Box p$

$\mathbf{5}: \Diamond\Box p \rightarrow \Box p$

$\mathbf{B}: \Diamond\Box p \rightarrow p$

$\mathbf{W5}: \Diamond\Box p \rightarrow (p \rightarrow \Box p)$

$\mathbf{F}: (p \wedge \Diamond\Box q) \rightarrow \Box(\Diamond p \vee q)$

some normal systems

Some well-known normal systems are denoted as follows

$B:\ \mathbf{B}$

$T:\ \mathbf{K}, \mathbf{T}$

$S4:\ \mathbf{K}, \mathbf{T}, \mathbf{4}$

$S4F:\ \mathbf{K}, \mathbf{T}, \mathbf{4}, \mathbf{F}$

$KD45:\ \mathbf{K}, \mathbf{D}, \mathbf{4}, \mathbf{5}$

$SW5:\ \mathbf{K}, \mathbf{T}, \mathbf{4}, \mathbf{W5}$

$S5:\ \mathbf{K}, \mathbf{T}, \mathbf{4}, \mathbf{5}$

Some relations between normal systems

- $K \subset T \subset S4 \subset S5$
- $K \subset B \subset S4 \subset S5$
- These logics are *sound* with respect to classes of models whose accessibility relations satisfy simple algebraic properties.

Some types of binary relations

Let R be a binary relation over a set X.

- R is *reflexive* if $R(a, a)$ for every $a \in X$
- R is *symmetric* if $R(a, b) \Rightarrow R(b, a)$ for every $a, b \in X$
- R is *transitive* if $R(a, b), R(b, c) \Rightarrow R(a, c)$ for every $a, b, c \in X$
- A relation that is reflexive, symmetric and transitive is said to be an *equivalence* relation.

Some soundness characterisations

- T is sound for the class of reflexive models, ie. the axiom $\mathbf{T} : \Box P \rightarrow p$ is valid in models whose R-relation is reflexive.

- $S4$ is sound wrt to models that are reflexive and transitive

- B is sound wrt models that are reflexive and symmetric.

- $S5$ is sound wrt models in which R is an equivalence relation.

soundness proofs

To prove soundness we must show

- The axioms of the system are true in all models of the given class
- The transformation rules US, MP and N are truth preserving, ie when applied to formulas true in all models, they lead to formulas true in all models.

soundness for K

so to prove soundness for the system K we must show

- The $\mathbf{K}$ axiom is true in all models. Given a model $\langle W, R, V \rangle$, it suffices to show that if (a) $\Box(p \rightarrow q)$ and $\Box p$ are true in a world w, then also (b) $\Box q$ is true in w. Suppose (a) holds. Then by (V $\Box$), $p \rightarrow q$ and p are true in all w' such that $R(w, w')$, so by (V $\rightarrow$) so is q. Therefore by (V $\Box$), $\Box q$ is true in w.

- The transformation rules US, MP and N are validity preserving, ie when applied to formulas true in all models, they lead to formulas true in all models. Suppose α is valid, then it is a formula true in every world w in any model. Then α is true independent of the truth-values assigned to the atomic variables in α. Hence if β is the result of uniformly replacing the variables of α by any wff, then β must also be true in w. So the rule US is validity preserving.

Exercises

- (1) show that the rules MP and N are validity preserving.
- (2) show that the axiom $\mathbf{T}$ is true in all reflexive models.
- (3) show that the axiom $\mathbf{B}$ is true in all reflexive, symmetric models.

preserving validity in a (single) model

The rule US of uniform substitution does not preserve truth in a single model. Counter-example: consider a model with two worlds w, w' with (w, w') as the only element in the R relation. Consider atoms p, q where p is true at both worlds and q at just the world w'. Then $\Box p \to p$ is true in the model but $\Box q \to q$ is not. Yet the latter is a substitution instance of the former.

However we do have the following:

- Theorem. Let S be an axiomatic, normal model system and let $\langle W, R, V \rangle$ be any model. If every substitution instance of every axiom of S is true in $\langle W, R, V \rangle$, then every theorem of S is true in $\langle W, R, V \rangle$.

- Note that the rules MP and N do preserve truth in a single model.

more on binary relations

Let R be a binary relation over a set X.

- R is *universal* if $R = X \times X$
- R is *Euclidean* if for every $a, b, c \in X$ such that $R(a, b)$ and $R(a, c)$, also $R(b, c)$
- Suppose $a \in X$ and there is no $b \in X$ such that $R(a, b)$, then a is called a *dead-end*.
- Note that if w is a dead-end, then we always have $V(\Box\alpha, w) = 1$ and $V(\Diamond\alpha, w) = 0$

completeness via canonical models

We want to show that certain classes of models *fully* characterise particular normal model systems. We use the powerful method of *canonical* models.

- Let S be a normal modal system and C a given class of models. A wff is said to be C-valid iff it is true in every model in C.

- S is *sound* wrt C if every theorem of S is C-valid.

- S is *complete* wrt C if every C-valid formula is a theorem of S; ie. if α is not a theorem of S ($\nvdash_S \alpha$) then it is not true in some C-model.

- A formula α is said to be S-*inconsistent* if $\vdash_S \neg\alpha$; otherwise (if $\nvdash_S \neg\alpha$) it is S-*consistent*. It follows that S is complete wrt C if $\forall\alpha$, if α is S-consistent then there is a C-model in which α is true at some world w.

maximal consistent sets

We first generalise S-consistency to sets of formulas

- Definition: A finite set $\Sigma = \{\alpha_1, \ldots, \alpha_n\}$ is S-consistent iff $\alpha_1 \wedge \ldots \wedge \alpha_n$ is S-consistent.
- An arbitrary set of formulas Σ is S-consistent if every finite subset of Σ is S-consistent, ie there is no finite $\{\alpha_1, \ldots, \alpha_n\} \subset \Sigma$ such that $\vdash_S \neg(\alpha_1 \wedge \ldots \wedge \alpha_n)$.
- the canonical model method will show that if Σ is an S-consistent set of wff, then there is a C-model $\mathcal{M} = \langle W, R, V \rangle$ and $w \in W$ such that $\mathcal{M}, w \models \Sigma$. $\mathcal{M}$ is called the canonical model.

maximal consistent sets, contd

- Definition: A set Γ of wff is maximal iff for every wff α, either $\alpha \in \Sigma$ or $\neg\alpha \in \Sigma$.

- Γ is said to be *maximal* S-consistent iff it is maximal and S-consistent.

  Lemma 1 Let Γ be a maximal S-consistent set of wff. Then:

  1. for any α, exactly one member of $\{\alpha, \neg\alpha\}$ is in Γ
  2. $\alpha \vee \beta \in \Gamma$ iff either $\alpha \in \Gamma$ or $\beta \in \Gamma$
  3. $\alpha, \beta \in \Gamma$ iff both $\alpha \in \Gamma$ and $\beta \in \Gamma$
  4. $\vdash_S \alpha \Rightarrow \alpha \in \Gamma$
  5. if $\alpha \in \Gamma$ and $\alpha \rightarrow \beta \in \Gamma$, then $\beta \in \Gamma$
  6. if $\alpha \in \Gamma$ and $\vdash_S \alpha \rightarrow \beta$, then $\beta \in \Gamma$

- **Theorem 2** Let Σ be S-consistent. Then there is a maximal S-consistent set $\Gamma \supseteq \Sigma$.

canonical models

First some notation: for Σ a set of wff, let $\Box^{-}(\Sigma) =_{df} \{\alpha : \Box\alpha \in \Sigma\}$

- **Lemma 3** let S be a normal system and Γ an S-consistent set of wff containing a wff of the form $\neg\Box\alpha$. Then $\Box^{-}(\Sigma) \cup \{\neg\alpha\}$ is S-consistent.

- Corollary. let S be normal and Γ an S-consistent set of wff containing a wff of the form $\Diamond\alpha$. Then $\Box^{-}(\Sigma) \cup \{\alpha\}$ is S-consistent.

- Definition. The *canonical* model of a normal modal system S is the model $\langle W, R, V \rangle$ defined as follows.

  1. $W = \{w : w$ is a maximal S-consistent set of wff$\}$
  2. For any $w, w' \in W$, $R(w, w') \Leftrightarrow \Box^{-}(w) \subseteq w'$.
  3. for any atom p and $w \in W$, $V(p, w) = 1 \Leftrightarrow p \in w$.

basic theorem for canonical models

- **Theorem 4** Let $\langle W, R, V \rangle$ be the *canonical* model of a normal modal system S. For any wff a and any $w \in W$, $V(\alpha, w) = 1 \Leftrightarrow \alpha \in w$.

- Proof. By induction on complexity of α

  1. For α an atom, claim holds by definition.
  2. Assume theorem for α and prove for $\neg\alpha$. Consider any $\neg\alpha$ and $w \in W$. By (V $\neg$), $V(\neg\alpha, w) = 1 \Leftrightarrow V(\alpha, w) = 0$. By assumption, $V(\alpha, w) = 0 \Leftrightarrow \alpha \notin w$. Hence $V(\neg\alpha, w) = 1 \Leftrightarrow \alpha \notin w$. By Lemma 1.1, $\alpha \notin w$ iff $\neg\alpha \in w$. Therefore $V(\neg\alpha, w) = 1 \Leftrightarrow \neg\alpha \in w$.
  3. For $\alpha \vee \beta$, assume claim holds for α and β, use (V $\vee$) and apply Lemma 1.2.
  4. Consider the case of $\Box\alpha$ and assume claims holds for α. (i) suppose $\Box\alpha \in w$. By definition of R, $\alpha \in w'$ for all w' such that $R(w, w')$. By induction assumption, for each such w', $V(\alpha, w') = 1$. So by (V $\Box$), $V(\Box\alpha) = 1$.

(ii) Suppose on the other hand that $\Box\alpha \notin w$. By Lemma 1.1, $\neg\Box\alpha \in w$. So by Lemma 3, $\Box^-(w) \cup \{\neg\alpha\}$ is S-consistent. Thus by Theorem 2 and definition of W, there exists a $w' \in W$ such that $\Box^-(w) \cup \{\neg\alpha\} \subseteq w'$. Hence we have (i) $\Box^-(w) \subseteq w'$ and (ii) $\neg\alpha \in w'$. (i) implies $R(w, w')$, by def of R. So by induction assumption, theorem holds for α and by part 1 above for $\neg\alpha$. Therefore by (ii), since $\neg\alpha \in w'$, we have $V(\neg\alpha, w') = 1$ and therefore $V(\alpha, w') \neq 1$. Then by (V $\Box$), we obtain $V(\Box\alpha) \neq 1$.

- Corollary. A formula α is valid in the canonical model for S iff $\vdash_S \alpha$. Proof: Let $\langle W, R, V\rangle$ be the *canonical* model for S. Suppose that $\vdash_S \alpha$. Then by Lemma 1.4, α belongs to every maximal S-consistent set. So $\alpha \in w$, for all $w \in W$. By Theorem 4, $V(\alpha, w) = 1$, for all $w \in W$ so α is true in the canonical model. Suppose that $\nvdash_S \alpha$. Then $\neg\alpha$ is S-consistent. So for some $w \in W$, $\neg\alpha \in w$ and hence $\alpha \notin w$. Therefore by Theorem 4, $V(\alpha, w) \neq 1$, for some $w \in W$, and so α is not true in the canonical model.

- Corollary. The system K is complete for the class of all models.

some completeness theorems

- T is complete with respect to the class of all reflexive models
- $S4$ is complete for the class of all reflexive, transitive models
- B is complete fore the class of all reflexive, symmetrical models
- $S5$ is complete for the class of all models in which R is an equivalence relation

Method Show in each case that the canonical model has the stated structure.

more completeness results

- D is complete with respect to the class of all models with serial accessibility relation

- $KD45$ is complete for the class of all transitive, Euclidean models with no dead-ends

- $S4F$ is complete fore the class of all reflexive, transitive models with the condition: if $R(a,b)$ and $R(a,c)$ but not $R(b,a)$, then $R(c,b)$.

- $SW5$ is complete for the class of all reflexive, transitive models in which R satisfies the condition: if $a \neq b, a \neq c$, $R(a,b)$ and $R(a,c)$, then $R(b,c)$ and $R(c,b)$.

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